The L'vov-Kaplansky Conjecture

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Noncommutative polynomials

Noncommutative polynomials are elements of the free noncommutative algebra over the field F

$$F\langle x_1, x_2, \dots\rangle = \left\{ \sum_i \lambda_i x_{j_{i1}} \cdots x_{j_{im_i}} \;\middle|\; \lambda_i \in F,\ j_{ik} \in \mathbb{N} \right\}.$$

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Images of polynomials

The *image* of a noncommutative polynomial $f(x_1, \dots, x_m)$ in the algebra A is

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- $(1 + xyx^2 - y^2z^3 + z)(A) = A$ surjective polynomial

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For an **infinite** field F and a non-central polynomial f, every traceless matrix is a sum of 2 matrices from the set $f(M_n(F)) - f(M_n(F))$.

Multilinear polynomials

Polynomials of the form

$$f(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} \lambda_\sigma x_{\sigma(1)} \ldots x_{\sigma(m)}$$

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Examples

- $\lambda xy + \mu yx$
- $g(x, y, z, w) = xyzw + 2wxzy - yxwz$
- $h(x, y, z) = xyz + zy$ is **not** multilinear

- If f is multilinear, then $f(A)$ is closed under multiplication with scalars.

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unknown for $n = 3$...
or polynomials of degree 4...

Proof Attempt (Dykema, Klep 2016)

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Sylwester's clock-and-shift matrices $(\omega^n = 1)$

$$u_q^p = \begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{bmatrix}^q \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}^p$$

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$$u_q^p \, u_{q'}^{p'} = \text{scalar} \cdot u_{q+q'}^{p+p'}.$$

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Now, try to solve $f(u_q^p, u_{q'}^{p'}, A) = B$...

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$$\det(x_1, \ldots, x_n, y_1, \ldots, y_n) \neq 0 \implies \text{solvable}$$

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- Quaternion algebras $\mathbb{H}_{\mathbb{R}}$.

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Theorem (V. 2021)

If A is an algebra with a surjective inner derivation, then every multilinear polynomial f is surjective in A.

Warring type problems

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Open problem

If f is non-central, is every traceless matrix contained in the set $f(M_n(F)) - f(M_n(F))$ for all big enough n?

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