A MAXIMAL NILPOTENT LINEAR SPACE CONTAINING NO MATRIX OF MAXIMAL NILINDEX

DANIEL VITAS

Abstract. Gutierrez Fernandez and Garcia conjectured that every maximal nilpotent linear subspace of $M_n(F)$ contains a matrix of nilindex n, and verified this for $n \leq 4$. We prove that the conjecture does not hold for $n = 5$ over every field of characteristic zero.

Let F be a field and n a positive integer. Denote by $M_n(F)$ the space of $n \times n$ matrices with entries in F. A linear subspace $\mathcal{V} \subseteq M_n(F)$ is called *nilpotent* if it consists of nilpotent matrices; a nilpotent linear subspace is *maximal* if it is not properly contained in another nilpotent linear subspace. The *nilindex* of a nilpotent matrix A is the least m with $A^m = 0$. The following conjecture is due to Gutierrez Fernandez and Garcia [1, Conjecture 1.1]; it is stated equivalently, in terms of rank, in [2, Conjecture 1].

Conjecture 1. *Every maximal nilpotent linear subspace of $M_n(F)$ contains a matrix of nilindex n.*

The conjecture was verified in [1] for $n \leq 4$. We show that it fails for $n = 5$.

Theorem 2. *Let F be a field of characteristic zero, and let $\mathcal{V}$ be the space of matrices of the form*

$$M(x,y,z,w) = \begin{pmatrix} 0 & 0 & 0 & z & 0 \\ 0 & 0 & x & -w & 0 \\ 0 & 0 & 0 & x+w & 0 \\ w & y & y-z & 0 & x \\ 0 & 0 & -y & z-y & 0 \end{pmatrix}$$

for $x, y, z, w \in F$. Then $\mathcal{V}$ is a maximal nilpotent linear subspace of $M_5(F)$, and $A^4 = 0$ for every $A \in \mathcal{V}$.

Proof. A simple calculation shows that

$$M(x,y,z,w)^4 = 0,$$

which proves that every element of $\mathcal{V}$ has nilindex at most 4; in particular, $\mathcal{V}$ consists of nilpotent matrices.

To show maximality, let $B = (b_{ij})$ be such that $\mathcal{V} + FB$ consists of nilpotent matrices; we will show that $B \in \mathcal{V}$. Since $\operatorname{char} F = 0$, by [3, Corollary 1] we have

$$\operatorname{tr}(M(x,y,z,w)^j B) = 0 \quad \text{for } j = 0, 1, 2, 3.$$

2020 *Mathematics Subject Classification.* Primary 15A30; Secondary 15A03.

Key words and phrases. Nilpotent matrices, nilpotent linear spaces, nilindex.

Collecting the coefficient of each monomial gives, for $j = 0, 1, 2$ and for four of the monomials with $j = 3$, the 19 equations

$$\begin{array}{rlrl}
 & j=0 & & \\
1: & b_{11}+b_{22}+b_{33}+b_{44}+b_{55}=0 & & \\
 & j=1 & & \\
x: & b_{32}+b_{43}+b_{54}=0 & y: & b_{24}+b_{34}-b_{35}-b_{45}=0 \\
z: & b_{41}+b_{45}-b_{34}=0 & w: & b_{14}-b_{42}+b_{43}=0 \\
 & j=2 & & \\
x^2: & b_{42}+b_{53}=0 & xy: & b_{23}+b_{33}-b_{45}-b_{55}=0 \\
xz: & b_{51}+b_{55}-b_{33}=0 & xw: & b_{13}+b_{42}-b_{52}+b_{53}=0 \\
y^2: & b_{25}+b_{35}=0 & yz: & b_{21}+b_{25}+b_{31}+2b_{35}=0 \\
z^2: & b_{31}+b_{35}=0 & zw: & b_{11}+b_{15}+b_{32}-b_{33}=0 \\
w^2: & b_{12}-b_{13}=0 & yw: & b_{15}+b_{22}-b_{23}+b_{32}-b_{33}+b_{45}=0 \\
 & j=3 & & \\
x^3: & b_{52}=0 & x^2y: & b_{22}+b_{32}-b_{55}=0 \\
x^2z: & b_{32}=0 & xyz: & b_{35}=0.
\end{array}$$

Solving this system of linear equations shows that

$$B=\begin{pmatrix} b_{51}+b_{55} & 0 & 0 & b_{54}-b_{53} & 0 \\ 0 & b_{55} & b_{45}-b_{51} & -b_{41} & 0 \\ 0 & 0 & b_{51}+b_{55} & b_{41}+b_{45} & 0 \\ b_{41} & -b_{53} & -b_{54} & -2b_{51}-4b_{55} & b_{45} \\ b_{51} & 0 & b_{53} & b_{54} & b_{55} \end{pmatrix},$$

where $b_{41}, b_{45}, b_{51}, b_{53}, b_{54}, b_{55}$ are the corresponding entries of B. Since B is itself nilpotent, we have $\operatorname{tr}(B^2) = \operatorname{tr}(B^3) = 0$; for the matrix above,

$$\operatorname{tr}(B^2) = 2\big(3b_{51}^2 + 10\, b_{51}b_{55} + 10\, b_{55}^2\big), \qquad \operatorname{tr}(B^3) = -6\,(b_{51}+2b_{55})\big(b_{51}^2 + 5\, b_{51}b_{55} + 5\, b_{55}^2\big).$$

It is easy to see that the only solution is $b_{51} = b_{55} = 0$, and therefore

$$B=\begin{pmatrix} 0 & 0 & 0 & b_{54}-b_{53} & 0 \\ 0 & 0 & b_{45} & -b_{41} & 0 \\ 0 & 0 & 0 & b_{41}+b_{45} & 0 \\ b_{41} & -b_{53} & -b_{54} & 0 & b_{45} \\ 0 & 0 & b_{53} & b_{54} & 0 \end{pmatrix} = M\big(b_{45},\, -b_{53},\, b_{54}-b_{53},\, b_{41}\big) \in \mathcal{V}. \qquad \square$$

Acknowledgments. The author is grateful to Matjaž Omladič, Heydar Radjavi, and Klemen Šivic for stimulating discussions and valuable insights concerning this work.

References

[1] J. C. Gutierrez Fernandez and C. I. Garcia, *Nilpotent linear spaces in* $M_4(F)$, Linear Multilinear Algebra **73** (2025), 3638–3651.

[2] J. C. Gutierrez Fernandez and C. I. Garcia, *Nilpotent linear spaces in* $M_5(\mathbb{C})$, Internat. J. Algebra Comput. **36** (2026), 249–283.

[3] B. Mathes, M. Omladič and H. Radjavi, *Linear spaces of nilpotent matrices*, Linear Algebra Appl. **149** (1991), 215–225.